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# TENTAMEN ALGEMENE RELATIVITEITSTHEORIE

Tuesday 01-02-2011, 14.00-17.00

On the first sheet write your name, address and student number. Write your name on all other sheets.

This examination consists of three problems, with in total 15 parts. The 15 parts carry equal weight in determining the final result of this examination.

$c = 1$.

## PROBLEM 1

Two observers, $A$ and $B$, are falling freely in orbits of constant $r$ in the Schwarschild-metric:

$$ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) .$$

The orbits are in the plane $\theta = \pi/2$, and have radius $r_A = 4m$, $r_B = 4^{4/3}m$. On $t = 0$ $A$ and $B$ pass through $\phi = 0$.

For circular, timelike geodesics with radius $r$ the following conditions hold:

$$(\dot{t})^2 = \frac{r}{r - 3m} = (\dot{\phi})^2 r^3/m ,$$

where the $\dot{}$ indicates differentiation with respect of the eigentime.

1.1 Calculate the coordinate time $\Delta t_A$ that $A$ needs for one orbit.

1.2 How much time does $A$ need according to his own clock for one orbit?

1.3 The clock of $B$ is lit and can be read by $A$ when they pass each other. What is the time difference that $A$ sees on the clock of $B$ between two consecutive passages of $B$ through the point $\phi = 0$?

1.4 How much time has passed on $A$'s clock while $B$ makes one orbit?

1.5 $A$ starts his rocket engine and stops at the point with coordinates $r = 4m$, $\theta = \pi/2$, $\phi = 0$. He then repeats the measurements of part (1.3) and (1.4). What are the results now?

**PROBLEM 2**

The Robertson-Walker metric for $k = 1$ can be written in the form

$$ds^2 = dt^2 - a(t)^2 \left\{ d\chi^2 + \sin^2\chi \left( d\theta^2 + \sin^2\theta d\phi^2 \right) \right\} .$$

Using the energymomentum tensor of an ideal fluid the Einstein equations imply the folowing relations between $a(t)$ and the functions $\rho(t)$ (energydensity) en $p(t)$ (pressure):

$$\frac{(\dot{a})^2 + 1}{a^2} = -\tfrac{1}{3}\kappa\rho \,,$$

$$\dot{\rho} + 3(p + \rho)\frac{\dot{a}}{a} = 0 \,.$$

The $\dot{}$ indicates differentiation with respect to $t$, $\kappa = -8\pi G$.

We consider a Friedmann universe with ultrarelativistic matter $\rho = 3p$.

2.1 Show that $\rho a^4$ is constant.

2.2 Determine $a$ as a function of $t$, with the boundary condition that at $t = 0$ we have $a = 0$.

2.3 Let $\rho_0$ en $a_0$ be the values of the functions $\rho$ and $a$ at time $t = t_0$. Show that this universe has a finite lifetime, and determine this lifetime as a function of $\rho_0$ and $a_0$.

2.4 Determine the trajectory of lightrays with $\dot{\theta} = \dot{\phi} = 0$.

2.5 A lightray is emitted at the origin of this universe at $t = 0$, $a = 0$, from the point with coordinate $\chi = 0$. What is the value of the coordinate $\chi$ when the value of $a$ is again zero?

**PROBLEM 3**

Consider a manifold $M$ with metric $g_{\mu\nu}$, and a symmetric metric connection $\Gamma$. Covariant derivatives will be indicated by semicolon (;) and ordinary derivatives by a comma (,). Covariant derivatives of vector fields take on the form:

$$(V^\mu)_{;\rho} \equiv V^\mu{}_{,\rho} + \Gamma^\mu_{\rho\nu} V^\nu \,, \qquad (V_\mu)_{;\rho} \equiv V_{\mu,\rho} - \Gamma^\nu_{\rho\mu} V_\nu \,,$$

where the connection $\Gamma$ is given by

$$\Gamma^{\rho}_{\mu\nu} \equiv \tfrac{1}{2} g^{\rho\sigma} \left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}\right) . \tag{3.1}$$

All covariant derivatives below are with respect to the connection (3.1).

3.1 Show that $g_{\mu\nu;\rho} = 0$.

3.2 On the manifold $M$ we have vector fields $A$ and $j$ satisfying Maxwell's equations:

$$F^{\mu\nu}{}_{;\nu} = j^\mu \tag{3.2}$$

where $F_{\mu\nu} = A_{\nu;\mu} - A_{\mu;\nu}$. Show that also

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} .$$

3.3 Define

$$\mathcal{F}^{\mu\nu} \equiv \sqrt{-g}\, F^{\mu\nu} , \quad \mathcal{J}^\mu \equiv \sqrt{-g}\, j^\mu ,$$

and $g = \det g_{\mu\nu}$. Show that the Maxwell equations (3.2) are equivalent to

$$\mathcal{F}^{\mu\nu}{}_{,\nu} = \mathcal{J}^\mu . \tag{3.3}$$

Use the fact that $\partial_\mu g = g\, g^{\rho\sigma}\, \partial_\mu g_{\rho\sigma}$.

3.4 Show that the Maxwell equations in the form (3.3) imply

$$\mathcal{J}^\mu{}_{,\mu} = 0 .$$

3.5 Use the result of (3.4) to show that $j^\mu{}_{;\mu} = 0$.



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